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THE PLANE CONTACT PROBLEM OF STEADY THERMOELASTICITY TAKING HEAT GENERATION INTO ACCOUNT†

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The plane steady contact problem of thermoelasticity when there is heat generation from friction, which arises when an infinite cylindrical punch moves over the surface of an elastic half-space along its generatrix, is considered. It is assumed that heat exchange between the free boundary of the half-space and the surrounding medium obeys Newton's law, while the condition for ideal thermal contact exists in the region in which the solids interact. The problem is reduced to a system of three integral equations in the heat fluxes and temperature. The effect of the thermal and mechanical properties of the cylinder and the half-space on the main contact characteristics is investigated numerically. © 1998 Elsevier Science Ltd. All rights reserved.

1. Suppose an elastic heat-conducting cylinder is pressed by a force P and slides along its generatrices with a constant velocity V over the surface of an elastic heat-conducting base (Fig. 1). As a result of the motion of the cylinder in the contact area $(-a, a)$, friction forces $\tau_{zy}(x)$ occur which obey Amonton's law and lead to heating of the rubbing solids. The heat generated is distributed between them depending on their properties and the contact conditions. Outside the contact area, heat exchange occurs between the solids and the external medium in accordance with Newton's law. It is assumed that when solving the thermoelasticity problem for a cylinder it can be replaced by a half-space. It is also assumed that the friction forces have no effect on the plane deformation of the half-space. The thermoelastic processes in the solids are considered to be steady.

All the characteristics which relate to the punch will be denoted by the subscript 1, while those relating to the half-space will be denoted by the subscript 2. With these assumptions it is required to determine the size of the contact area, the distribution of the contact pressure, the heat fluxes, and also the temperature fields in the contacting solids.

In mathematical terms, the problem reduces to solving the equations of steady thermoelasticity

$$2(1-\nu_i)\frac{\partial^2 u_i}{\partial x^2}+(1-2\nu_i)\frac{\partial^2 u_i}{\partial y^2}+\frac{\partial^2 \upsilon_i}{\partial x\,\partial y}=2(1+\nu_i)\,\alpha_i\frac{\partial T_i}{\partial x}$$

$$(1-2\nu_i)\frac{\partial^2 \upsilon_i}{\partial x^2}+2(1-\nu_i)\frac{\partial^2 \upsilon_i}{\partial y^2}+\frac{\partial^2 u_i}{\partial x\,\partial y}=2(1+\nu_i)\,\alpha_i\frac{\partial T_i}{\partial y}$$

$$\frac{\partial^2 T_i}{\partial x^2}+\frac{\partial^2 T_i}{\partial y^2}=0 \quad (i=1,2) \tag{1.1}$$

for the punch ($i = 1$) and for the half-space ($i = 2$) with the following boundary conditions on the line $y = 0$

$$q_1(x)+q_2(x)=-V\sigma_{zy}(x),\quad T_1(x)=T_2(x)=T(x),\quad |x|<a \tag{1.2}$$

$$(-1)^i K_i\partial T_i/\partial y+h_iT_i(x)=0,\quad |x|>a \tag{1.3}$$

$$\sigma_{zy}(x)=fp(x),\quad |x|<a \tag{1.4}$$

$$\sigma_{yy}^{(i)}(x)=0,\quad |x|>a,\quad \sigma_{xy}^{(i)}(x)=0,\quad |x|<\infty \tag{1.5}$$

$$d(\upsilon_1(x)-\upsilon_2(x))/dx=-x/R,\quad |x|<a \tag{1.6}$$

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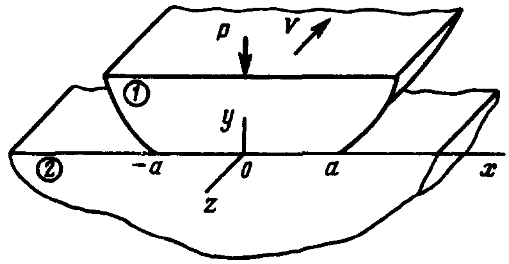

Fig. 1.

Here u_i and υ_i are the components of the displacement vector, $\sigma_{yy}^{(i)}$, $\sigma_{xy}^{(i)}$, $\sigma_{zy}^{(i)}$ are the components of the stress tensor, T_i is the temperature, q_i is the heat flux and $p(x)$ is the contact pressure. In addition, ν_i, μ_i, α_i, K_i, h_i are Poisson's ratio, the shear modulus, the coefficient of linear thermal expansion, the thermal conductivity and the heat-transfer coefficient for the punch ($i = 1$) and the half-space ($i = 2$), respectively, f is the friction coefficient and R is the radius of curvature of the punch base.

2. Applying a Fourier transformation with respect to the variable x to the solution of problem (1.1), (1.2), (1.5) we obtain that the surface temperatures of the contacting solids satisfy the integral equations

$$T_i(x) - \frac{h_i}{\pi K_i}\int_{-a}^{a} T_i(x')\,M_i(x-x')\,dx' = \frac{1}{\pi K_i}\int_{-a}^{a} q_i(x')\,M_i(x-x')\,dx'$$

$$M_i(z) = \int_0^{\infty} \frac{\cos(\xi z)}{\xi + h_i / K_i}\,d\xi \tag{2.1}$$

while the derivatives of the vertical thermoelastic displacements of the surfaces of these solids along the horizontal axis have the form

$$(-1)^{i+1}\frac{d\upsilon_i}{dx}\frac{1-\nu_i}{\pi\mu_i}\int_{-a}^{a}\frac{p(x')}{x-x'}\,dx' + \frac{\delta_i}{\pi}\int_{-a}^{a}[q_i(x') + h_i T_i(x')]\,N_i(x-x')\,dx'$$

$$N_i(z) = \int_0^{\infty}\frac{\sin(\xi z)}{\xi + h_i / K_i}\,d\xi,\quad \delta_i = (1+\nu_i)\frac{\alpha_i}{K_i} \tag{2.2}$$

We note two limiting cases

(a) $h_i = 0$ (thermally insulated surfaces); letting the coefficients h_i in (2.1) and (2.2) tend to zero, we obtain the well-known results [1, 2]

$$T_i(x) = \frac{1}{\pi K_i}\int_{-a}^{a} q_i(x')\,M_0(x,x')\,dx',\quad M_0(x,x') = -\ln|x-x'| + c_i \tag{2.3}$$

$$(-1)^{i+1}\frac{d\upsilon_i}{dx} = \frac{1-\nu_i}{\pi\mu_i}\int_{-a}^{a}\frac{p(x')}{x-x'}\,dx' + \frac{\delta_i}{2}\int_{-a}^{a} q_i(x')\,\mathrm{sign}(x-x')\,dx' \tag{2.4}$$

where c_i = const;

(b) $h_i \to \infty$; conditions (1.3) then have the form

$$T_i(x) = 0,\quad |x| > a$$

and the solution of the problem, obtained by the method of dual integral equations, is given by relations similar to (2.3) and (2.4) with

$$M_0(x,x') = \mathrm{Arsh}\,\frac{2a^2 - x^2 - x'^2}{|x^2 - x'^2|} - \mathrm{Arsh}(1) \tag{2.5}$$

In the two limiting cases of the problem the derivatives of the vertical displacements are identical.

By means of relations (2.2) we can satisfy boundary condition (1.6), using in this case conditions (1.2) and (1.4) and converting relations (2.1) taking the second condition of (1.2) into account, and we then change to the dimensionless quantities

$$x = as,\quad x' = ar,\quad p(x) = \frac{P}{a}p^*(s),\quad T(x) = \frac{fPV}{K_1 + K_2}T^*(s)$$

As a result we obtain a system of three integral equations in terms of the temperature and heat fluxes (we omit the asterisks)

$$\frac{1}{\pi}\int_{-1}^{1} q_1(r)\left[\frac{1}{s-r} - \frac{2f\,\mathrm{Pe}\,H}{1+\delta}N_1(s-r)\right]dr + \frac{1}{\pi}\int_{-1}^{1} q_2(r)\left[\frac{1}{s-r} - \frac{2\delta f\,\mathrm{Pe}\,H}{1+\delta}N_2(s-r)\right]dr -$$

$$-\frac{2f\,\mathrm{Pe}\,H}{\pi}\int_{-1}^{1} T(r)\left[\frac{\mathrm{Bi}_1}{(1+K)(1+\delta)}N_1(s-r) + \frac{\delta K\,\mathrm{Bi}_2}{(1+K)(1+\delta)}N_2(s-r)\right]dr =$$

$$= \frac{2a^2}{\pi a_{\text{H}}^2}\bar{P}s,\quad |s|<1 \tag{2.6}$$

$$T(s) - \frac{\mathrm{Bi}_i}{\pi}\int_{-1}^{1} T(r)\,M_i(s-r)\,dr - \frac{1+K}{\pi\varkappa_i}\int_{-1}^{1} q_i(r)\,M_i(s-r)\,dr = 0$$

We need to solve the system of integral equations obtained with the condition of equilibrium of the punch, which, taking the first condition of (1.2) into account, can be written in the dimensionless form

$$\int_{-1}^{1} q_1(r)\,dr + \int_{-1}^{1} q_2(r)\,dr = 1 \tag{2.7}$$

Here

$$\mathrm{Pe} = \frac{Va}{2k_1},\quad \mathrm{Bi}_i = \frac{h_i a}{K_i},\quad \varkappa_1 = 1,\quad \varkappa_2 = K,\quad K = \frac{K_2}{K_1},\quad \delta = \frac{\delta_2}{\delta_1}$$

$$H = K_1(\delta_1 + \delta_2)\left(\frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2}\right)^{-1},\quad \bar{P} = \frac{P_{\text{H}}}{P}$$

where K_i are the thermal diffusivities of the solids.

The half-length of the contact area a_{H} and the pressing force P_{H} in the corresponding Hertz problem are related by the equation [3]

$$a_{\text{H}}^2 = \frac{2P_{\text{H}}R}{\pi}\left(\frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2}\right)$$

The kernels of the system of integral equations have the form

$$M_i(z) = \int_0^\infty \frac{\cos\zeta z}{\zeta + \mathrm{Bi}_i}d\zeta,\quad N_i(z) = \int_0^\infty \frac{\sin\zeta z}{\zeta + \mathrm{Bi}_i}d\zeta$$

To evaluate these we will use the representations in terms of the integral sine and cosine [4]

$$M_i(z) = -\mathrm{Ci}(|z|\,\mathrm{Bi}_i)\cos(z\mathrm{Bi}_i) - \mathrm{si}(|z|\,\mathrm{Bi}_i)\sin(|z|\,\mathrm{Bi}_i)$$

$$N_i(z) = [\mathrm{Ci}(|z|\,\mathrm{Bi}_i)\sin(|z|\,\mathrm{Bi}_i) - \mathrm{si}(|z|\,\mathrm{Bi}_i)\cos(z\,\mathrm{Bi}_i)]\,\mathrm{sign}(z)$$

The kernels $N_i(z)$ are regular while $M_i(z)$ have a logarithmic singularity.

The contact area is also unknown in the system of integral equations (2.6). It can be determined by iteration, checking that condition (2.7) is satisfied. Here we used a more simple method: the half-length of the contact area was specified (for example, $a/a_H = 1$, which corresponds to the isothermal contact area), and from system (2.6), (2.7) the ratio $\bar{P}$, necessary to maintain a specified contact area, was determined. This approach was used in [2].

After solving the system of integral equations (2.6) and (2.7) the dimensionless contact pressure is found from the relation

$$p(s) = -[q_1(s) + q_2(s)], \quad |s| < 1$$

Assuming the half-space to be absolutely rigid ($\mu_2 \to \infty$, $\delta_2 = 0$) and non-heat-conducting ($K_2 = 0$), we obtain from (2.6) and (2.7) a system of two equations of the problem for a single solid [5].

3. It can be seen from the system of integral equations (2.6) and (2.7) that the operator on the heat-flux function is a Cauchy operator of the first kind, while the operator on the temperature is a Fredholm operator of the second kind with a logarithmic kernel. We will therefore represent the heat fluxes in the form

$$q_i(s) = \varphi_i(s)(1 - s^2)^{1/2} \tag{3.1}$$

where $\varphi_i(s)$ are regular functions, while the temperature will be sought in the space of bounded functions.

To discretize the system of integral equations obtained, we will use the Gauss–Chebyshev quadrature formulae [6] for the integrals with heat fluxes and the trapezium quadrature formulae for the integrals with temperature. As a result we obtain a closed system of linear algebraic equations for determining the ratio $\bar{P}$, and also the values of the required functions $\varphi_i(s)$ and $T(s)$ at discrete points of the contact area. The system obtained is solved on a computer.

4. The input parameters of the problem are the Biot coefficients: Bi_1 and Bi_2, the ratios δ and K, and also the complex characteristic $f\mathrm{Pe}H$. Instead of the quantity δ it is more convenient to consider the parameter

$$\varkappa = (1 + \delta K)/[(1 + K)(1 + \delta)]$$

the role of which will be indicated below.

To study the thermal contact between the solids an important quantity is the coefficient of the distribution of the heat fluxes between the solids

$$q = \frac{1}{2}\int_{-1}^{1} \frac{q_1(x) - q_2(x)}{q_1(x) + q_2(x)}\, dx$$

In Fig. 2 we show this quantity as a function of the parameter K for fixed $\mathrm{Bi}_1 = 1$ for different values of the coefficient Bi_2. When $\mathrm{Bi}_1 = \mathrm{Bi}_2$ we obtain from the system of integral equations (2.6)

$$q = (1 - K)/(1 + K)$$

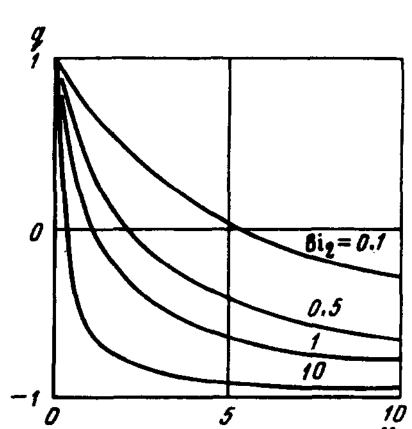

Fig. 2.

Figure 2 shows that an increase in the heat exchange on the free surface of one of the solids leads to an increase in the heat flux entering the same solid. Calculations enable us to choose the combination of the parameter K and the Biot coefficients to obtain the necessary heat-flux distribution between the solids.

The temperature distribution in the contact area when $\mathrm{Bi}_1 = \mathrm{Bi}_2$ is shown in Fig. 3(a). An increase in the heat generation leads to a considerable reduction in the temperature level at the contact. The dashed curve corresponds to the case when $\mathrm{Bi}_1, \mathrm{Bi}_2 \to \infty$ and shows the temperature distribution calculated using (2.3) and (2.5). Note that in the other limiting case when $\mathrm{Bi}_1 = \mathrm{Bi}_2 = 0$ the temperatures in the solids may be calculated from (2.3) only after specifying an additional condition defining the constants c_i. The temperature distribution when $\mathrm{Bi}_1 = \mathrm{Bi}_2$ is independent of K. If the Biot constants are different: $\mathrm{Bi}_1 = 1$, $\mathrm{Bi}_2 = 0.1$ (Fig. 3b), the temperature distribution depends on the ratio of the thermal conductivities of the solids K. In view of the symmetrical form of the temperature it is only given on one half of the contact area.

When solving the problem we also determine the ratio $\bar{P}$, which enables us to compare the size of the contact area obtained with the corresponding solution of the Hertz problem. For a single solid, when the surface outside the contact area is thermally insulated, it was shown in [7] that the contact area is less than in the corresponding Hertz problem. Moreover, it was found that there is a critical value of the parameter $f\mathrm{Pe}H$ for which $\bar{P} = 0$ (i.e. it is necessary to apply an infinitely large force P in order to obtain a contact area that is the same as in the problem without heat generation). The numerical analysis carried out here enables us to investigate the effect of heat generation on $\bar{P}$, and of course, on the contact area.

Note that there are combinations of the input parameters of the problem for which the heat generation has no effect on the size of the contact area. One of these cases is obtained when $K = 0$ and $\varkappa = 1$, which corresponds to the contact between an elastic heat-conducting cylinder and a rigid thermally insulated base.

The second case arises for all values of K when $\mathrm{Bi}_1 = \mathrm{Bi}_2$. Then the ratio $\bar{P}$ falls linearly as $f\mathrm{Pe}H$ increases and, moreover, depends only on the parameter

$$\bar{P} = 1 - \varkappa f \mathrm{Pe}\, H / [f \mathrm{Pe}\, H]_0$$

where $[f\mathrm{Pe}H]_0 = 1.16$ is the critical value of the parameter $f\mathrm{Pe}H$, obtained in [7] for the case of a single thermoelastic solid. The value $\bar{P}$ decreases as $\varkappa$ increases, and consequently, the contact area also decreases.

In Fig. 4 we show $\bar{P}$ as a function of $\varkappa$ for fixed values of $f\mathrm{Pe}H = 0.1$ and $\mathrm{Bi}_1 = 1$ for different values of Bi_2. It can be seen that when $\varkappa = 0.5$ the ratio $\bar{P}$ is also independent of the heat generation, where this occurs for all values of K.

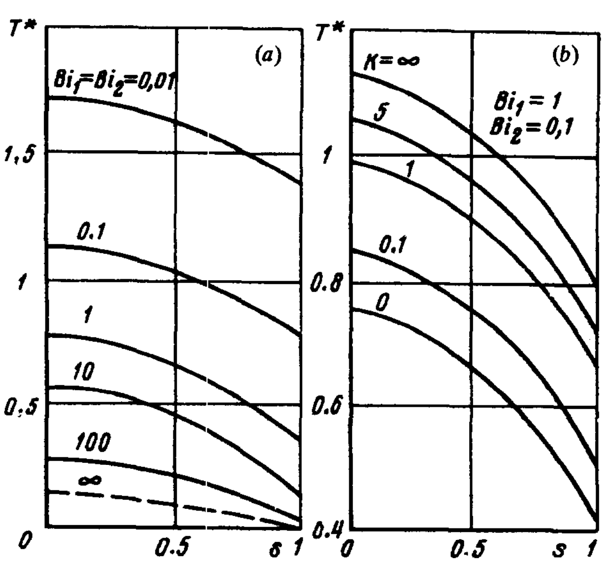

Fig. 3.

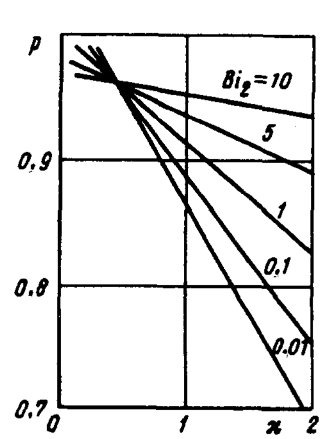

Fig. 4.

In general, the ratio $\bar{P}$ depends linearly on $f\mathrm{Pe}H$

$$\bar{P} = 1 - f\,\mathrm{Pe}\,H\,/\,[f\,\mathrm{Pe}\,H]$$

where the critical value $[f\,\mathrm{Pe}\,H]$ is a function of the coefficients Bi_i. The effect of hat generation on the value of $[f\,\mathrm{Pe}\,H]$ is given below for two values of $\varkappa$ and $K = 0.5$

Bi_2	0.01	0.1	1.0	5.0	10.0
$[f\,\mathrm{Pe}\,H]$ for $\varkappa = 2.0$	0.33	0.41	0.57	0.84	1.13
$[f\,\mathrm{Pe}\,H]$ for $\varkappa = 0.4$	3.64	3.25	2.86	2.63	2.52

When obtaining these results the value of one of the Biot parameters was fixed ($\mathrm{Bi}_1 = 1$) and the parameter Bi_2 was varied. When $\varkappa = 2$ an increase in the heat generation on the surface of one of the solids leads to an increase in $[f\,\mathrm{Pe}\,H]$ and, of course, to an increase in $\bar{P}$ and of the contact area. In the other case ($\varkappa = 0.4$) the opposite effects occur. This phenomenon is due to the fact that the parameter $\varkappa$ passes through the value $\varkappa = 0.5$.

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